Effects of a flexible boundary on hydrodynamic stability

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The theoretical study presented in this paper was inspired by the recent report (Krämer 1960) of experiments showing that considerable reductions in the drag of an underwater solid body were achieved by covering it with a skin of flexible material; apparently this effect was due to the boundary layer being stabilized in the presence of the skin, so that transition to a turbulent condition of flow was prevented or at least delayed. The stability problem for flow past a flexible boundary is here formulated in a general way which allows a full exploration of the possibility of a stabilizing effect without the need to assign specific properties to the flexible medium; the collective properties of possible boundaries are represented by a 'response coefficient' α (a sort of 'effective compliance') measuring the deflexion of the surface under a travelling sinusoidal distribution of pressure.

A remarkably simple analytical connexion is established between the present general problem and the corresponding stability problem for the boundary layer on a rigid plane wall, and hence many details of the existing theory of hydrodynamic stability are immediately useful. However, the presence of the flexible boundary admits possible modes of instability additional to those which already exist when the boundary is rigid, and clearly every mode must be considered with regard to practical measures for stabilization—that is to say, it might be useless to inhibit one mode by a device which lets in another. What is believed to be an essentially complete interpretation of the over-all possibilities is deduced on recognizing three more or less distinct forms of instability. The first comprises waves resembling the unstable waves which can arise in the presence of a rigid boundary, but now being modified by the effects of flexibility. These waves tend to be stabilized when the boundary has a compliant response to them, which means the respective wave velocity has to be less than the velocity of *free* surface waves on the boundary; but it is found that the effect of internal friction in the flexible medium is actually *de*stabilizing. The second form of instability is essentially a resonance effect and comprises waves travelling at very nearly the velocity of free surface waves. These waves can only be excited when the latter velocity falls below the free-stream velocity; they are scarcely affected by the viscosity of the fluid since the 'wall friction layer' is largely cancelled, so that damping due to the medium itself becomes the only stabilizing factor. The third form is akin to Kelvin–Helmholtz instability.

This interpretation of the theoretical results seems to point to the essential factors in the operation of a flexible skin as a stabilizing device, and accordingly in the concluding section of the paper two alternative sets of criteria are proposed

each of which would provide a logical basis for designing such a device. The principle of the first alternative explains the success of Kramer's invention, but the second appears equally promising and the relative advantages of the two can really be proved only by further experiment.

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1. Introduction

The recently reported experiments by Krämer (1960*a*, *b*) have evidently aroused wide interest in the possible uses of coatings of elastic material as a practical means of preventing transition to turbulence in laminar boundary layers. His important pioneering papers presented evidence that the device invented by him had brought about considerable reductions in the drag of underwater bodies, and following the publication of his original idea there has naturally been speculation regarding other applications of this method of stabilization. The purpose of the present paper is to examine this idea theoretically, attempting to reveal the general practical possibilities rather than to establish complete solutions applicable to specific stabilizing devices. The problem is treated by use of the Tollmien–Schlichting type of stability theory, and the discovery is made that the minimum Reynolds number for the existence of neutral wave disturbances is in various circumstances increased by flexibility of the boundary. This is encouraging from a practical point of view, because in its more familiar applications, the perturbation theory of boundary-layer stability has been well-confirmed experimentally.

At first sight the present problem appears vastly more complicated than the corresponding one for a rigid boundary, which itself is by no means easy; but a way is found to adapt the results of previous theories fairly readily, so that solutions can be found without very much additional labour. The main difficulty intrinsic to all stability problems of the present kind is to solve the hydrodynamical equations to an adequate degree of approximation; but there is no need to tackle this anew since our problem admits use of the 'Tietjens function' and other functions depending on the general solution of the Orr–Sommerfeld equations which have been calculated previously. The analysis in fact leads to an equation which, except for a simple additional term, is the same as the central result of Lin's modified formulation (1945, 1955) of the Tollmien–Schlichting theory for parallel flows with rigid boundaries. As in this previous case, the complete class of neutral stability conditions is represented by this equation; in the present case, however, its interpretation is rather more difficult.

The proposed method of stabilization may generally require close specification to insure a useful result, since its application is liable to have the contrary effect of destabilizing the flow. The extra mobility introduced when a flow boundary is made flexible creates the possibility of modes of instability additional to those which may already exist when the boundary is rigid, and there is one such type of instability whose importance in present respects can at once be recognized. This takes the form of waves progressing in the flow direction at very nearly the same speed as free surface waves in the flexible medium, the waves being amplified by the action of the flow which supplies sufficient energy to counterbalance internal dissipation. The mechanism of generation of surface waves in this way has been studied at length by Miles (1957) and Brooke Benjamin (1959: hereafter this paper will be referred to as I). As formulated in the following pages, the stability analysis covers this type of waves as well as a type that can be considered to comprise Tollmien–Schlichting waves modified by the influence of the flexible boundary. The two types are not really distinct analytically, but the physical problem is most readily understood by considering them separately. (Two such distinct classes of waves were recognized by Lock (1954) in his treatment of the stability problem—which is closely related to the present one—for a boundary layer at the interface between an air stream and deep water; he called them respectively 'water waves' and 'air waves'. The theoretical results presented in his paper are very complicated, and are insufficient to indicate a general physical interpretation of the effects of the water surface on stability.) A third type, which will be called 'Kelvin–Helmholtz' instability in view of its analogy in the classical problem of discontinuous fluid motions, has also to be recognized as a possibility.

The analysis will be made on the usual lines of stability theories for parallel flows, and many of the familiar details will be taken for granted. We generally take dimensionless variables according to the usual scheme (Schlichting 1955, p. 316), but we change without comment to dimensional forms whenever this is helpful and the new meaning is clear. The case of boundary layers with positive (adverse) pressure gradient will not be covered explicitly; this case is within the scope of the theory, and the general conclusions derived here would appear to apply to it, but it presents complications which can well be left for later study. The essential effects to be brought to light are demonstrated adequately with reference to boundary layers with negative or zero pressure gradient, for which the velocity profile has no inflexion. Most of the analysis is concerned with neutral disturbances which are simple-harmonic in time and in the co-ordinate x parallel to the primary flow velocity $U(y)$; that is, all velocity and stress perturbations have the common factor $\exp\{ik(c-ct)\}$, where the wave-number k and wave-velocity c are real. The justification for confining attention to two-dimensional disturbances will be considered briefly in §5.

As a preliminary to the main analysis in §3, an outline of the Tollmien–Schlichting theory is given in §2. The results recalled in §2 are all quite well known (e.g. see Schlichting 1955, ch. 16, or Lin 1955), but it is a great advantage to have a short account of them here for easy reference in the later parts of the paper.

2. The Tollmien–Schlichting theory of boundary-layer stability

The essential points of this theory are to be adapted directly to the present problem, and its physical basis is specially worth reconsidering here. The theory recognizes that, for the large Reynolds numbers at which neutral small disturbances become possible, the effects of viscosity are confined to two 'friction layers' whose thickness is very much smaller than that of the whole boundary layer. The first adjoins the wall and its thickness is $O(kR)^{-\frac{1}{2}}$. The second surrounds the critical point $y = y_c$ at which $U = c$, and this layer is rather more diffuse than the first, its thickness being $O(kR)^{-\frac{1}{3}}$. A vital step in the theory is the assertion that an adequate approximation to the solution of the Orr–Sommerfeld equation can be found in the form

$$\phi(y)+f(y), \tag{2.1}$$

where $\phi(y)$ is the solution of the inviscid version of the equation, and $f(y)$ is a rapidly varying solution which can be taken to be insignificant outside the wall layer: $f(y)$ is approximated as the solution of another simplified version of the equation derived from the facts that $f''/f = O(kR) \gg U''/c$ and $\gg k^2$. The condition to be satisfied by the solution at the outer edge of the boundary layer is taken to be a condition on ϕ alone; thus,

$$\phi'/\phi \to -k \quad \text{for} \quad U \to U_0. \tag{2.2}$$

The effects of viscosity over the inner friction layer are not explicitly represented in this approximate solution, and the solution is therefore invalid in this region. However, the theory demonstrates that (2.1) will be a uniformly valid approximation on either side of the inner friction layer, provided that in $y < y_c$ the inviscid solution ϕ takes on a certain feature which is left unresolved by the inviscid equation. The critical point is a singular point of this equation, and consequently ϕ' has a logarithmic singularity at $y = y_c$. In passing through $y = y_c$, therefore, ambiguity arises as to which branch of the logarithm to take. Tollmien showed how the choice is determined from the complete Orr–Sommerfeld equation; he investigated an approximate solution which tends to ϕ for $y > y_c+O(kR)^{-\frac{1}{3}}$ but which remains valid in the immediate region of $y = y_c$, and so indicates the appropriate form of ϕ for $y < y_c-O(kR)^{-\frac{1}{3}}$. When the approximation (2.1) is adjusted according to the principle established by Tollmien and substituted in the boundary conditions at the wall $y = 0$, relations between the parameters k, R, c can be found which describe the conditions of neutral stability.

The conditions of zero normal and tangential velocity at the wall require that

$$\left.\begin{aligned} \phi_w+f_w &= 0,\\ \text{and}\qquad \phi'_w+f'_w &= 0, \end{aligned}\right\} \tag{2.3}$$

where the subscript denotes values for $y = 0$. Hence

$$\frac{f_w}{f'_w} = \frac{\phi_w}{\phi'_w}. \tag{2.4}$$

Now, f_w/f'_w can be found from the simplified equation describing the effects of viscosity in the wall layer, and the only characteristic of the boundary-layer profile on which it depends is the initial gradient U'_w. When equation (2.4) is multiplied by $-U'_w/c$, the quantity given on the left-hand side is found to be a function of $z = (kRU'_w)^{\frac{1}{3}}\, c/U'_w$ alone; and the right-hand side, being independent of R, is a function of k and c only. Thus the equation can be expressed as

$$F(z) = E(k,c). \tag{2.5}$$

The 'Tietjens function' $F(z)$ has been extensively tabulated; and since it is independent of the form of the velocity profile, it is universally applicable to stability problems of the present kind. The function $E(k, c)$ depends on the overall velocity profile, and the calculation of this function is the most difficult part of the practical task of finding neutral-stability curves. Later in this paper use is made of an approximate expression for $E(k, c)$, such as was used originally by Tollmien and Schlichting to estimate this function for the Blasius velocity profile on a flat plate at zero incidence. More accurate calculations have been made by Lin (1945, 1955), Holstein (1950), Shen (1954) and other writers, and their results are suitable for adaptation to our problem; but the expression we shall consider has the advantage of simplicity, and seems adequate for the purpose of this exploratory study.

For details of the calculation of neutral-stability curves by the original method, we may refer to Schlichting's book (1955, pp. 327–329). The imaginary parts of F and E are plotted against their real parts on the same graph, the curves being determined parametrically by z, k and c. From the points of intersection of the curves, one obtains sets of values of z, k, c representing conditions of neutral stability. Elimination of c from z then gives the co-ordinates of the neutral-stability curve in the $(R,\ k)$-plane.

This outline of the theory omits the various refinements due to Lin and others which have led to improved estimates of the neutral-stability curves. For instance, the Tietjens function can be applied in a more exacting way than at present (see Lin 1955, pp. 40, 80), although the simple definition is reasonably adequate for most cases of interest—more specifically, for cases where the critical point is close to the boundary, so that the approximations $U'_w \doteqdot U'_c$ $(\equiv U'(y_c))$, $y_c \doteqdot c/U'_w$ are fairly good. However, the main improvements that have been made on the original calculations of Tollmien and Schlichting are due to better estimates of the function $E(k, c)$, and the status of this function in the modern theories remains the same as in the original theory. The important thing to note in this connexion is that although we shall introduce here some rather crude approximations taken from the Tollmien–Schlichting theory, the analysis of the flexible-boundary problem is set in a form such that the way is clear to take advantage of the best available results if the need for precise data should arise.

One small detail of post-Schlichting theories needs to be noted. This is a modification of the formula (2.5) which was introduced by Lin (see Lin 1955, p. 40). If one defines

$$\mathscr{F}(z) = \frac{1}{1-F(z)}, \quad u+iv = \frac{1}{1-E},$$

(2.5) can be put in the form

$$\mathscr{F}(z) = u+iv. \tag{2.6}$$

This alternative form has the practical advantage that successive approximations to $u+iv$ can be expressed much more conveniently than the equivalent approximations to E. It turns out that v varies only slightly with k (see Lin 1955, bottom of p. 40), and this fact is found to be highly significant in the present problem.

The physical ideas underlying the theory, particularly with regard to the two friction layers, provide a useful insight into the analytical results derived later; and although the points in question are fairly well known, they deserve to be reconsidered here as a preliminary to our physical interpretation of the effects of a flexible boundary. It is a remarkable fact, first realized clearly by Prandtl many years ago, that viscosity is responsible for the instability of laminar boundary layers in the absence of adverse pressure gradient. For fluids without viscosity, two theorems due to Rayleigh prove that a parallel flow whose velocity profile has no inflexion is stable; and so, if for a real fluid such a flow is to be unstable at all, viscosity must be capable of a destabilizing action in addition to its expected damping effect on small disturbances. Prandtl (1935) showed that the required destabilizing effect is generated by the wall friction layer. When the primary flow (at large yet finite R) is given a wavy disturbance, this layer of intense vorticity occurs because the tangential velocity component of the disturbance has to be brought to zero right at the wall. The vorticity distribution in the layer is such as to displace the effectively inviscid flow outside it in a manner quite unlike that which would happen if viscosity were entirely absent and the edge of the flow were governed by a kinematical condition alone. The velocity components u, v of the disturbance just outside the wall layer are thereby given a difference in phase such that the average Reynolds stress $-\rho\overline{uv}$ is positive. This means that energy is converted from the basic flow into the disturbance, at a rate $-\rho\overline{uv}U'\delta y$ per unit length in each layer δy; and when the energy supply is sufficient to balance viscous dissipation, a neutral disturbance can occur. It is a simple matter to show that this Reynolds stress is given by

$$\tau = \tfrac{1}{2}\rho k \mathscr{I}(\phi^*\phi') = -\tfrac{1}{2}\rho k \mathscr{I}(\phi\phi'^*),$$

where the asterisk denotes complex conjugates and $\mathscr{I}$ the imaginary part.

As a slight variation on Prandtl's argument, we observe from (2.4) that the phase relation between ϕ and ϕ' right at the wall is determined by properties of the viscous solution f. But whereas f becomes insignificant outside the wall layer, the values of ϕ and ϕ' for a considerable distance beyond the layer are very approximately the same as their values at the wall. Hence, putting $\phi_w = (f_w/f'_w)\,\phi'_w$, we conclude that a close approximation to the Reynolds stress outside the wall layer is

$$\tau = -\tfrac{1}{2}\rho k\,|\phi'_w|^2\,\mathscr{I}(f_w/f'_w) = \tfrac{1}{2}\rho k(c/U'_w)|\phi'_w|^2\,\mathscr{I}\{F(z)\}. \tag{2.7}$$

The imaginary part of the Tietjens function is positive for all relevant values of z (z being necessarily positive for waves travelling in the flow direction), except for small values ($< 2{\cdot}3$) at which the critical point is brought so close to the wall that the two friction layers overlap and the present assumption of an intermediate inviscid region breaks down. Thus, the Reynolds stress is shown to be positive, as it must be for a neutral disturbance to be maintained against the dissipative action of viscosity.

The theory of the inviscid equation shows that $\mathscr{I}(\phi^*\phi')$ is constant throughout any range of y not including the critical point; but on opposite sides of the critical point the constant can have different values. The boundary condition (2.2) requires that ϕ is in phase with $-\phi'$ at the outer edge of the boundary-layer profile, which means that the Reynolds stress is zero there and hence, because of the property noted in the last sentence, it must be zero everywhere beyond the inner friction layer. The phase difference necessary for the positive Reynolds stress in the region between the two friction layers is therefore acquired entirely at the critical point, this effect being of course just another interpretation of the matter discussed in the second paragraph of this section.

Tollmien was the first to establish an explicit relation between the Reynolds stress and conditions at the critical point (see Lin 1955, p. 54). His result, applicable to the neutral case, is

$$\tau = -\pi\rho k\,|\phi_c|^2\, U''_c/U'_c. \tag{2.8}$$

This formula confirms that τ can be positive, since the critical point obviously can occur where the profile curvature U''_c is negative.

3. Theory for a flexible boundary

Consider first the boundary condition to be satisfied at $y = 0$ if the boundary is flexible, though still solid, and a wave travelling at velocity c is superposed on it. The equation of the deformed boundary may be written

$$y = a\,e^{ik(x-ct)}$$

with the amplitude a considered to be in general a complex quantity. If this wave is generated by the action of the flow alone, as we shall presently suppose, all velocity and stress perturbations in the fluid are proportional to a; however, it is important not to express these perturbations with a as a common factor, because the mathematical problem has to be formulated in such a way that the Tollmien-Schlichting problem is recovered by letting $a \to 0$. For the wavy boundary, the kinematical and non-slipping conditions on the fluid velocity components take the following form (cf. I, p. 169), where the subscript w again denotes values at $y = 0$:

$$\left.\begin{aligned} \phi_w + f_w &= ca, \\ \phi'_w + f'_w &= -U'_w a. \end{aligned}\right\} \tag{3.1}$$

These are the exact boundary conditions according to linearized theory, the only restriction on their validity being that $|a|$ should be very much smaller than the wavelength, so that $k|a| \ll 1$. Condition (2.2) applies as before at the outer edge of the boundary layer, which means that ϕ_w/ϕ'_w is exactly the same function of k and c that occurs in the Tollmien–Schlichting analysis.

We now assume that the wave on the boundary arises in response to a pressure fluctuation developed in the fluid. The effect of the accompanying shear-stress fluctuation can be neglected, since this stress is extremely small in comparison with the pressure (see I, §7). Expressing the surface pressure distribution by

$$p_s = \rho P_s e^{ik(x-ct)},$$

we can represent the response of the flexible medium by the equation

$$a = \alpha P_s/U'_w c, \tag{3.2}$$

which defines a parameter $\alpha/U'_w c$ dependent only on the properties of the medium and on k and c. The case of a rigid boundary is represented by $\alpha \to 0$. (The factor $1/U'_w c$ is included in this definition merely for convenience later.)

A very good approximation to P_s can be obtained in terms of ϕ_w and ϕ'_w by using the fact, demonstrated in I, that the pressure varies to a quite insignificant extent across the wall friction layer. The result is (cf. I, §3)

$$P_s = k^2 \int_0^\infty (U-c)\,\phi\,dy = U'_w \phi_w + c\phi'_w. \tag{3.3}$$

By use of (3.2) and (3.3) the boundary conditions (3.1) can now be arranged to give

$$\left.\begin{aligned}(1-\alpha)\,\phi_w - \frac{\alpha c}{U'_w}\,\phi'_w + f_w &= 0,\\ (1+\alpha)\,\phi'_w + \frac{\alpha U'_w}{c}\,\phi_w + f'_w &= 0.\end{aligned}\right\}$$

Hence, introducing the functions $F(z)$ and $E(k,c)$ defined in §2, we obtain directly

$$\begin{aligned}F(z) &= \frac{E+\alpha(1-E)}{1+\alpha(1-E)}\\ &= E_1(k,c), \quad \text{say.}\end{aligned} \tag{3.4}$$

This result has the same standing as the Tollmien–Schlichting formula (2.5), and clearly reduces to it when $\alpha \to 0$, i.e. when the boundary is made rigid. When the properties of the flexible medium are specified, α can be found as a function of k and c; and so, just as with (2.5), the right-hand side of (3.4) is calculable as a function of k and c only, while the left-hand side is again a function of z alone. Neutral-stability curves can therefore be found by the same graphical method as before.

A more convenient way of treating the problem appears, however, when F and E are expressed in terms of the functions $\mathscr{F}$ and $u+iv$ used in Lin's formula (2.6). Equation (3.4) then gives

$$\mathscr{F}(z) = u + iv + \alpha. \tag{3.5}$$

Thus, a remarkably simple analytical connexion is established between the present case and the case of a rigid wall. Nevertheless, it does not follow that the *complete* solution of our problem is provided merely by a simple transformation of the neutral-stability curves for a rigid wall, because the additional term α entering Lin's formula cannot be prescribed physically other than as a function of k and c, without restriction on the range of these parameters, and so the equation may admit solutions quite unrelated to those for $\alpha = 0$. In particular, we must allow the possibility of neutral waves of the kind studied by Miles (1957) and in I, which are largely independent of viscosity.

Three aspects of the above result will be considered under separate headings as follows. It may be helpful to note in advance that this division of material coincides roughly with our coverage of the expected three classes of waves, but there is necessarily some overlapping.

Non-dissipative flexible media

In this case α is real, being positive or negative accordingly as the wave velocity c is greater or less than the velocity of free waves on the boundary (see below). We first consider a property of the transformation on the right-hand side of

(3.4) which is discovered when the values of E_r and E_i calculated by Schlichting for the Blasius profile are substituted; we refer to the plot of E_i *vs* E_r given in his book (1955, p. 328). For α real, it appears that a plot of the real and imaginary parts of our function $E_1(k, c)$ describes exactly the same lines c = const. as the plot of E_i *vs* E_r. In other words, the complex value of E_1 is given everywhere as the value of E for the same c but some different value of k; thus $E_1(k, c) = E(k_0, c)$, say. This property is unexpected; but the reason for it is soon found when the nature of the approximations in Schlichting's calculation is examined, and the property is seen to be only an approximate one.

As in many existing stability calculations, Schlichting's estimate of E was essentially a first-order approximation for small k. It was equivalent to the following approximation to $u+iv$ noted by Lin (1955, p. 86):

$$u+iv = 1+U'_w c\int_0^1 \frac{dy}{(U-c)^2}+\frac{U'_w c}{k(1-c)^2}. \tag{3.6}$$

The path of integration extends from the wall to the outer edge of the boundary layer ($y = 1$ since the boundary-layer thickness is implied to be the unit of length), and is indented *under* the singularity of the integrand at $U(y_c) = c$ on the real axis of y. Hence, by use of the calculus of residues, (3.6) gives

$$u = 1+U'_w c\mathscr{P}\int_0^1 \frac{dy}{(U-c)^2}+\frac{U'_w c}{k(1-c)^2}, \tag{3.7}$$

$$v = -\pi c U''_c/U'^2_c, \tag{3.8}$$

where $\mathscr{P}$ denotes the principal value of the integral. According to this approximation, v is independent of k. It follows that if z, k_0, c is a set of values which satisfies (3.5) when $\alpha = 0$, then the equation is also satisfied, for $\alpha \neq 0$ and real, with the same z and c but with a k such that $u(k, c)+\alpha = u(k_0, c)$. That is,

$$\frac{U'_w c}{(1-c)^2}\left\{\frac{1}{k}-\frac{1}{k_0}\right\}+\alpha = 0. \tag{3.9}$$

At this point it would clearly be an advantage to redefine α without the normalizing factor $U'_w c$ introduced in (3.2), and hence cancel this factor from (3.9). As an alternative we define a 'stiffness coefficient' for the flexible medium, writing

$$\beta = -\frac{P_s}{ka} = -\frac{U'_w c}{k\alpha}. \tag{3.10}$$

An important advantage of this definition is that, anticipating considerations made below regarding the possible properties of the flexible medium, we recognize that the functional dependence of β on k and c always follows the form $\{c_0^2(k)-c^2\}F(k)$; and for a non-dispersive frictionless medium (see §4), we have always $\beta = Ak\{c_0^2-c^2\}$, where A and c_0 are constants independent of k. In terms of β, (3.9) gives immediately

$$k = k_0\left\{1-\frac{(1-c)^2}{\beta}\right\}. \tag{3.11}$$

This simple result provides us readily with a complete description of the effect of a non-dissipative flexible boundary on the stability of Tollmien–Schlichting waves. The conclusions to be based on this result are, of course, only approximate; but they would appear to be fairly reliable since more accurate estimates of v show it to vary only very slightly with k, which is the crucial property in present respects.

To fix ideas, let us reconsider Schlichting's graph of the imaginary versus the real parts of $F(z)$ and $E(k, c)$ for the Blasius profile. It has been established just above that we can use this graph—or any similar ones for other profiles—exactly as it stands, if the lines k = const. are now interpreted as k_0 = const. Therefore, from each point of intersection of the F and E curves, we get a set of values of

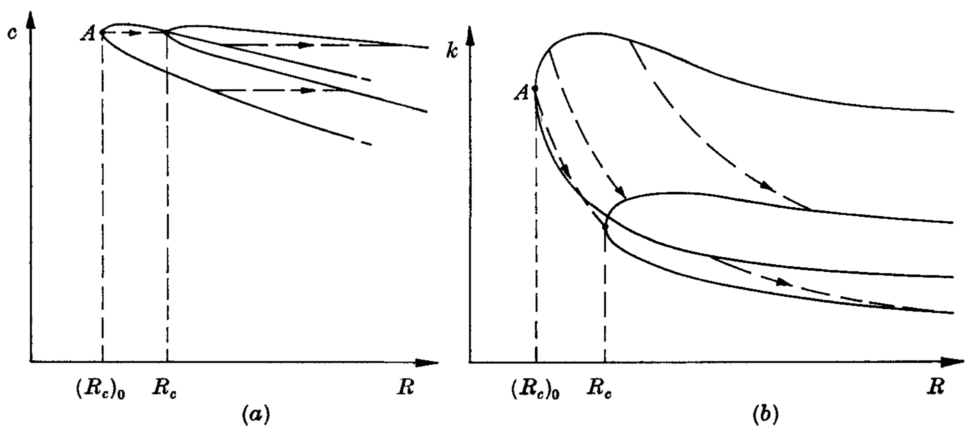

FIGURE 1. Displacements of curves of neutral stability due to a compliant non-dissipative boundary ($\alpha_r < 0$, $\alpha_i = 0$).

z, k_0, c which is the same set of values specifying neutrally stable conditions in the case of a rigid boundary. But, from the definition of z, the product kR must be the same for each given pair of values of z and c; thus kR for a flexible boundary is the same as $k_0 R_0$, where R_0 is the Reynolds number deduced from the same point of the graph in the rigid case. This means simply that the neutral curve c *vs* R will be displaced from the corresponding curve for a rigid boundary (see Schlichting 1955, p. 329) in the manner indicated in figure 1*a*: to the right if $\alpha < 0$ (i.e. $\beta > 0$) so that according to (3.11) $k_0 > k$, and to the left if $\alpha > 0$ so that $k_0 < k$. Points on the neutral curve k *vs* R will be displaced along the hyperbolae $kR = k_0 R_0$ as indicated in figure 1*b*: downward if $\alpha < 0$, and upward if $\alpha > 0$.

It is clearly possible that in a practical example $\alpha(k, c)$ changes sign within the relevant range of k and c, so that different parts of the neutral curves are displaced in opposite directions. When this possibility is examined with regard to general properties of flexible media, an important new feature of the problem appears. In fact, according to the present theory for α real, the flow is destabilized down to indefinitely small Reynolds numbers when α can change its sign at any c within the range of Tollmien–Schlichting waves, and further study shows this to be true when α changes sign at any positive $c < 1$. The reason is as follows.

Whatever the actual nature of the non-dissipative mechanical system forming a boundary to the flow, it will have the property that, in the absence of the flow, simple-harmonic surface waves are transmitted at a certain velocity c_0—which depends on k if the system is 'dispersive'. This assertion implies no more than that the system has both inertia and stiffness. Now, to have $\alpha < 0$, which means that an applied surface pressure is in opposite phase to the wave formed on the boundary in response to it, the velocity c has to be less than c_0 for the respective k, because the stiffness of the system is then the predominant factor in determining the response (cf. the example treated in § 4). If the pressure is to be in phase with the wave so that $\alpha > 0$, we must have $c > c_0$ because the inertia of the system is then the predominant factor. In fact, as has already been noted, it turns out generally that for a fixed k the effective stiffness β is proportional to $c_0^2 - c^2$. This result bears out the already fairly obvious conclusion that, if c approaches c_0, there is a resonance effect and the response of the system becomes unbounded in the limit. It follows that if equation (3.5) admits a solution with $(c - c_0) \to 0^+$ for any k, the neutral curve will have a loop extending to the region $R \to 0$.

Thus, if an effectively frictionless flexible device were to be used for stabilization, one essential requirement appears to be that its characteristic surface-wave velocity $c_0(k)$ should not be less than the greatest velocity of neutral Tollmien–Schlichting waves, which for the Blasius profile is about 0·42 times the free-stream velocity.

Consider now point A of figure 1a and 1b which gives the critical Reynolds number—i.e. the minimum value for which a neutral disturbance is possible—for a rigid boundary. Assuming that this point is transformed approximately to a corresponding extreme point on the neutral curve given by a real negative α, which is true if α does not vary too rapidly with k and c, we have

$$R_c = \frac{k_0}{k}(R_c)_0, \tag{3.12}$$

where $(R_c)_0$ is the original critical value, k_0 is the corresponding wave-number, and $k\ (< k_0)$ is the solution of

$$k = k_0\left\{1 - \frac{(1-c_1)^2}{\beta(k, c_1)}\right\}, \tag{3.13}$$

which is equation (3.11) with the velocity c_1 of the optimum Tollmien–Schlichting wave substituted. (For the Blasius profile, we have $c_1 \doteqdot 0{\cdot}41$.) By means of similar relationships, other points (k, R) on the neutral curve could be derived from the co-ordinates of the original curve. From these results it is clearly a simple matter to estimate the critical Reynolds number when $\beta(k, c)$ is specified for a particular flexible device.

According to these results, an obvious requirement for a large stabilizing effect is that β should be small for small k. Although for a non-dispersive medium (see § 4), β is proportional to k, this idealized case is an unhelpful example in present respects since (3.13) has no meaningful solution; furthermore, it can be seen from the discussion of Kelvin–Helmholtz instability a few paragraphs below that if this property were specified to hold down to indefinitely small k, this second form of instability would appear to be inevitable if $c_0 < 1$, which seems unrealistic. However, for any real flexible medium the coefficient β is bound to become large at finite wavelengths λ, since for indefinitely great length scales it is impossible to keep the ratio P_s/a decreasing as rapidly as $1/\lambda$; in fact, the practical extreme is that P_s/a tends to a constant (i.e. a uniform compressibility). In practice, therefore, β will have a certain minimum, and for effective stabilization this should evidently be made to occur at a reasonably small k, preferably rather smaller than the typical k_0 of Tollmien–Schlichting waves. We now assume this to be the case, and since the actual extent of stabilization then depends on the magnitude of the minimum rather than on the particular k, we can usefully write $\beta(k, c) = \beta_m(c)$ in the theory and ignore the variations of β with k in so far as they fix the solutions of (3.11) or (3.13).

The interpretation of (3.13) presents certain difficulties which are to be discussed below in connexion with Kelvin–Helmholtz instability, and it will appear that stabilization to indefinitely high Reynolds number, which (3.13) indicates to be a possibility, may not in fact be realizable. However, we can recognize a wide range of conditions where there is a considerable stabilizing effect on Tollmien–Schlichting waves while at the same time the possibility of other forms of instability is ruled out. For example, suppose that the minimum stiffness under static loading ($c = 0$) is $\beta_m(0) = 1$, and also $c_0 = 1^+$. Kelvin–Helmholtz instability cannot occur under these conditions (see equation (3.17) below). Since

$$\beta_m(c) = \beta_m(0)\left\{1-\frac{c^2}{c_0^2}\right\},$$

we get from (3.13), for $c_1 = 0{\cdot}41$,

$$\frac{k}{k_0} = 1-\frac{(0{\cdot}59)^2}{1-(0{\cdot}41)^2} \doteqdot 0{\cdot}58.$$

Hence, according to (3.12), the critical Reynolds number is increased by approximately 72 %, which implies that the distance from the leading edge at which instability first occurs is increased by a factor $(1{\cdot}72)^2 \doteqdot 3{\cdot}0$.

It remains to consider whether other types of neutral wave are possible, such perhaps as to specify the limit of an additional mode of instability in circumstances where Tollmien-Schlichting waves are damped. The case where $c \doteqdot c_0$ has obviously not yet been properly covered; for even if c_0 is made greater than the velocity of any Tollmien–Schlichting wave, so that the apparently disastrous effect on stability described above is avoided, waves with speeds near c_0 may still be possible.

The other outstanding case is as follows.

Kelvin–Helmholtz instability

Equations (3.12) and (3.13) indicate that the optimum Tollmien–Schlichting wave would be stabilized to extremely high Reynolds number if $\beta_m(c_1) \rightarrow (1-c_1^2)$, and similar results apply to waves with other values of c. This interpretation must be treated with caution, however, because values of β which are of the order of unity imply that the surface stiffness is of the same order of magnitude as the dynamic pressure of the free stream, and in such circumstances the Kelvin–Helmholtz type of instability (see Lamb 1932, §§232, 268) becomes an important possibility. This mechanism of instability is largely independent of viscosity, and theoretical treatments of it (see particularly Miles 1959) generally assume an inviscid fluid; the leading result of the theory as regards the flow properties were in fact justified in the paper I as asymptotic limits at infinite Reynolds number. This does not, of course, mean that this type of instability occurs only at very large Reynolds number; to the contrary, the fact that it can happen in the absence of viscosity implies that when the physical situation is such as to be more than marginally unstable at infinite Reynolds number, the instability will still be manifested at quite low Reynolds numbers since a large viscous effect is necessary to overcome the destabilizing factor.

The present analysis is inappropriate to account directly for Kelvin–Helmholtz instability, but a slight reformulation of ideas is enough to explain it. At very large Reynolds number the pressure on a wave is very nearly of opposite phase to the surface displacement, although there remains a small component proportional to k which is in phase with the wave slope (I, §7); provided k is small enough for the latter component to be insignificant, a neutral wave is therefore possible when the negative in-phase pressure just balances the stiffness forces in the flexible medium, which tend to cancel the wave. Hence, using a result derived in I (equation (7.34)), we deduce the condition for a neutral wave to be

$$\beta(k,c) = \frac{P_s}{ka} = \int_0^\infty (U-c)^2 e^{-ky} d(ky). \tag{3.14}$$

If k is very small, $U = 1$ over most of the range in which $\exp(-ky)$ is significant, so that (3.14) gives very approximately

$$\beta(k,c) = (1-c)^2. \tag{3.15}$$

Thus we have an interpretation of the result indicated by (3.11) or (3.13) that a neutral wave exists at infinite Reynolds number when (3.15) is satisfied.

Considering the minimum value of β, we have

$$\beta_m(c) = \beta_m(0)\left\{1-\frac{c^2}{c_0^2}\right\}, \tag{3.16}$$

which is a result that has already been noted, and on substitution of this into (3.15) a quadratic equation for c is obtained. Kelvin–Helmholtz instability is indicated by this equation having complex conjugate roots (cf. Lamb, §232). The instability condition is found to be

$$\beta_m(0) < 1-c_0^2 \quad (c_0 < 1), \tag{3.17}$$

and the velocity of the wave which first becomes unstable is $c = c_0^2$.

We shall not pursue this topic further since for present purposes it seems enough to note (3.17) as a practical test for this type of instability. Nevertheless, the connexion between the present results and the asymptotic results for Tollmien–Schlichting waves is a matter of great theoretical interest, and certainly it needs further study if our general problem is to be fully resolved.

Dissipative flexible media

In this case α has a negative imaginary part which, as will be verified later, is a measure of the work that must be done by the pressure on the disturbed boundary in order to maintain a neutral wave.

For a given z, equation (3.5) shows v to be greater by an amount $-\alpha_i$ than its value in the case of a rigid boundary. But, according to the approximation (3.8), v is a function of c alone that increases with increasing c over all values relevant to Tollmien–Schlichting waves. For $\alpha_i < 0$, therefore, wave velocities are greater than the velocities of the respective neutral waves for a rigid boundary. It turns out also that the values of k have to be slightly large, for the increased values of c, in order to satisfy the real part of (3.5) for a given z.

When z is fixed, R is proportional to $k^{-1}c^{-3}$, and hence we see that a negative imaginary part of α tends strongly to shift the neutral stability curves everywhere towards lower Reynolds numbers. If $-\alpha_i$ is large enough, it may even cancel the stabilizing effect of a negative α_r. This general conclusion is confirmed when one applies the transformation (3.4) with complex α to Schlichting's results for E.

It is thus established that dissipation in the flexible medium has a *destabilizing* effect in relation to Tollmien–Schlichting waves. This result is surprising at first sight, but its physical explanation is readily forthcoming when Prandtl's and Tollmien's interpretations of the mechanism of instability—as recalled in § 2—are applied in the present context. We shall return to this matter presently.

We consider next the class of waves on the boundary whose velocity of propagation is determined mainly by the properties of the flexible medium; i.e. these are essentially 'free' waves with c very nearly equal to c_0. Unlike the cases previously considered, the essential physical factor now is that the over-all stiffness of the boundary is quite high (e.g. for static loading, $\beta(k, 0) \gg 1$), and the response to the flow depends primarily on the resonance effect at $c \doteqdot c_0$. Thus the Kelvin–Helmholtz type of instability is altogether ruled out, since the effect of the negative in-phase pressure will displace c only very slightly below c_0.

As regards the interpretation of (3.5) in this case, the outstanding fact is that the equation represents essentially a balance between $u+iv$ and α, owing to the fact that α_r is now extremely sensitive to the value of c. Thus the effects of viscosity as represented by the function $\mathscr{F}(z)$ ceases to have a critical influence on the conditions for a neutral wave, and to represent typical neutral conditions this function can well be ignored. (Of course, if we specified k, R and $c \doteqdot c_0$ to be typical of Tollmien–Schlichting waves, this function has the same sort of magnitude as previously considered; but the important thing now is that the question of whether the wave is stable or unstable under such conditions depends critically on factors other than viscosity, and so these conditions are not representative of neutral waves.) We therefore take the neutral conditions to be given typically by $u = -\alpha_r, v = -\alpha_i$, which are equivalent to

$$E_r = 1 + \left(\frac{1}{\alpha}\right)_r, \tag{3.18}$$

$$E_i = \left(\frac{1}{\alpha}\right)_i. \tag{3.19}$$

Now, although these two equations have an unfamiliar form, a little study shows that they reproduce essentially the problem treated by Miles (1957) and further investigated in I. Miles derived a theory of water-wave formation by wind on the assumptions that (i) the water-wave velocity is independent of the air flow, (ii) the conditions for a neutral wave may be decided very critically by approximate energy considerations, i.e. an expression for the energy transfer from the air flow is put equal to an expression for the rate of viscous dissipation in the water, and (iii) the effects of viscosity in the air are negligible, except in so far as they determine the phase change across the 'critical layer' at which $U = c$ (see § 2). It is now seen that a similar theory would account for waves of the present sort.

The first fact to be observed from (3.18) and (3.19) is that $E = -U'_w\phi_w/c\phi'_w \doteqdot 1$ very nearly, which means simply that the boundary conditions (3.1) are satisfied by ϕ alone, i.e. there is no wall friction layer. This would be the natural starting point if the present special case were to be treated individually in a way such as Miles used. Actually the non-slipping boundary condition cannot be applied rigorously to an inviscid flow; but, as was discussed in I, this condition is satisfied automatically if one applies the modified kinematical condition introduced by Miles—i.e. $\phi \to (c-U)\,a$ for $y \to 0$, which means physically that streamlines close to the boundary must follow the wave contour as well as the bounding streamline itself. The fact that the latter condition is only approximate allows the small difference $(1/\alpha)$ from the result $E = 1$ which would follow if both the kinematical and non-slipping boundary conditions were applied rigorously to the inviscid solution. The essential scheme of Miles's theory is to find ϕ from the approximate boundary condition and then estimate E_i, as needed in (3.19), from properties of ϕ accumulated over the whole flow.

The term $(1/\alpha_r)$ in (3.18) is evidently of little consequence. It merely implies that the wave velocity is not quite c_0, since the in-phase negative pressure on the wave makes a slight reduction in the over-all stiffness of the system as experienced by the wave.

The crucial relation with regard to stability conditions is (3.19). This can be shown to correspond exactly to Miles's criterion that the energy supply to a neutral wave on the boundary balances the rate of energy absorption. Thus we recognize that the stability of waves of the present sort depends critically on the internal damping of the flexible medium. Instability can always be prevented by making the damping large enough, although we know that damping has a contrary effect on the mode of instability examined previously. Note that these waves can only be excited when c_0 is less than the free-stream velocity, since only then is E_i (or v) positive; i.e. a positive Reynolds stress is developed by a wave only when there is a critical point at which $U(y_c) = c \doteqdot c_0 < 1$.

As a final comment on the analysis, in particular to interpret the remarkable difference in the effects of damping noted just above, we note that the imaginary part of (3.5) is a precise representation of the energy balance maintained by a neutral wave; thus we have

$$v - |\alpha_i| - \mathscr{F}_i(z) = 0, \tag{3.20}$$

where the three terms represent the relative magnitudes of, respectively, the rate of energy conversion from the primary flow by Reynolds stress, the work done on the boundary by the surface pressure distribution, and the rate of dissipation by viscosity. [It is fairly easy to check this result. For instance, the rate of energy conversion τc may be estimated by Tollmien's formula (2.8); and hence by use of the approximations $\phi_c = \phi_w + (c/U'_w)\,\phi'_w$ and $U'_c = U'_w$, it may be identified with the approximation (3.8) for v; again a precise identification with v may be established by taking the formula (2.7) for τ and using the boundary conditions. The second term in (3.20) is readily established from the fact that the average work done on the boundary is $-\overline{p_s\,\partial\eta/\partial t}$, where η is the displacement.]

For the surface waves that were studied last, the relative energy supply v is essentially fixed since c is nearly equal to c_0 which is a property of the flexible medium. Thus we see that internal damping, as represented by $|\alpha_i|$ can take over the role that viscosity would have in stabilizing a neutral wave. For the modified Tollmien–Schlichting waves, on the other hand, the over-all stiffness must be so small in order to have any effect that c is not fixed by the properties of the medium. Thus an increase in internal damping is readily accommodated by an increase in v (i.e. the flow is permitted to develop additional Reynolds stress); and from our knowledge of the structure of the wall friction layer (i.e. its dependence on the special combination of parameters in z), we see that for an increased c the Reynolds number has to be reduced in order to make the relative viscous dissipation sufficient to restore the energy balance for a neutral wave.

4. The flexible medium

It has been seen that a requirement for any useful stabilizing device is that it comprises a *dispersive* medium for the transmission of surface waves. The precise analysis of such devices is liable to be fairly complicated, and it is not proposed to consider any example here. The case of a non-dispersive medium is very straightforward, however, and the following simple treatment is included to illustrate the meaning of the response coefficient α. Two instances of non-dispersive media, i.e. media in which free waves are propagated at a velocity independent of wavelength, are (i) a flexible inextensible sheet under longitudinal tension and (ii) a deep layer of uniform elastic material, the surface waves in this case being 'Rayleigh waves' (Love 1927, §214).

Consider a small displacement $y = \eta(x,t)$ of the plane surface bounding the medium. In the absence of dissipative forces, the dynamical equation satisfied by η is the wave equation

$$T\frac{\partial^2\eta}{\partial x^2} - m\frac{\partial^2\eta}{\partial t^2} = 0. \tag{4.1}$$

Here T is a constant 'stiffness coefficient' for the surface, and m is the effective mass per unit area of the surface. In the case of a flexible sheet, T is simply the tension per unit span and m the actual mass per unit area; but these constants have corresponding, though perhaps less immediate, interpretations in all examples of the class considered. The velocity $\pm c_0$ of free waves is therefore given by

$$c_0^2 = T/m. \tag{4.2}$$

When the waves are subject to linear damping, their equation takes the modified form

$$c_0^2\frac{\partial^2\eta}{\partial x^2}-\frac{\partial^2\eta}{\partial t^2}-\kappa\frac{\partial\eta}{\partial t}=0, \tag{4.3}$$

where κ is a damping coefficient, being necessarily real and positive. (The medium is now dispersive, but only very slightly so if κ/kc_0 is small; for example, non-periodic waves are propagated with little change of form though with over-all attenuation.)

Now, (4.1) and (4.3) are in effect equations relating the normal force per unit area developed on the surface $y = 0$ to the inertial reaction of the medium exerted across this surface. And so, if an external pressure $p_s(x, t)$ is applied on this surface, the equation governing the response of the medium clearly is

$$p_s=m\left(c_0^2\frac{\partial^2\eta}{\partial x^2}-\frac{\partial^2\eta}{\partial t^2}-\kappa\frac{\partial\eta}{\partial t}\right). \tag{4.4}$$

For applied pressure $\quad p_s=\rho P_s e^{ik(x-ct)}$

producing a deflexion $\quad \eta = a\,e^{ik(x-ct)},$

equation (4.4) shows that

$$\rho P_s = m\{(c^2-c_0^2)\,k^2+i\kappa kc\}\,a. \tag{4.5}$$

Hence, for the surface compliance a/P_s, we get

$$\frac{a}{P_s}=\frac{\rho}{m}\left\{\frac{(c^2-c_0^2)\,k^2-i\kappa kc}{|(c^2-c_0^2)\,k^2+i\kappa kc|^2}\right\}, \tag{4.6}$$

and α is the dimensionless coefficient obtained on multiplying this by cU'_w. An electrical analogue of this result is the 'effective capacity' of a series *LCR* circuit; the pressure P_s is analogous to an applied voltage alternating at frequency kc, and the displacement a is analogous to the resulting charge on the condenser.

Two points illustrated by this result may be noted. For the real part of α to be negative, which was shown in § 3 to promote stabilization, we must have $c_0^2 > c^2$; i.e. the velocity of free waves in the medium must exceed the velocity of the relevant flow disturbances. This criterion for a negative α_r in fact applies generally, whether or not c_0 depends on k, and its physical basis is obvious. (It is perhaps already perfectly clear from the analogy noted above.) Consider the wave which travels over the surface of the medium in response to a simple-harmonic pressure distribution with a certain k and c. If $c^2 < c_0^2$, the frequency $|kc|$ of the forced vibrations at a fixed point of the surface is less than the frequency $|kc_0|$ at which an undamped free simple-harmonic motion can occur, and at which therefore the inertial forces generated in the motion exactly balance the restoring forces due to the stiffness of the medium. Thus, the stiffness forces will predominate when $c^2 < c_0^2$, so that the surface will respond to the applied pressure in the same direction as it would to a static load. Equation (4.6) also confirms that for positive c the imaginary part of α is negative when the medium is dissipative, i.e. when energy is absorbed from the agency responsible for the moving pressure distribution.

5. Conclusion

The theoretical results derived in § 3 serve at least to point out the essential factors in the operation of a flexible boundary as a stabilizing device, and so they may be useful as a guide for further experimental development of Krämer's idea (1960*a*, *b*). It must be emphasized, however, that the present theory only concerns stability with respect to very small disturbances to which the principle of linear superposition can be applied. Nothing has been learnt about the influence of a flexible boundary on flow disturbances which are not small, although in relation to the practical problem this matter may be just as important as the one studied here. That is to say, it remains in question whether a flexible boundary might inhibit the development of the concentrated turbulent spots which are often the first manifestation of instability in laminar boundary layers when the flow has a high level of 'background' turbulence.

The stability analysis dealt only with long-crested waves travelling in the direction of flow, yet 'three-dimensional' waves travelling obliquely to the flow are obviously a physical possibility. For parallel flows with rigid boundaries, it is well known that consideration of two-dimensional waves is sufficient to solve the stability problem since Squire's theorem (Lin 1955, § 3.1) shows these waves to have the greatest tendency to instability. The same theorem applies to our problem also, though the conclusions to be drawn from it are not quite as definite as in the previous case. If the response of the boundary is assumed to be independent of the wave direction, as it well might be in practice, then the stability problem for an oblique wave is the same as that for a wave in the flow direction at a lower flow velocity; this follows from the fact that an oblique wave depends on the primary flow only in respect of its component in the direction of propagation of the wave. As regards the modified Tollmien–Schlichting waves and the Kelvin–Helmholtz waves which have been considered, the present two-dimensional theory would appear therefore to account adequately for the practical stability conditions. This may not always be so, however, as regards surface waves whose velocity in any direction of approximation is approximately c_0, for a situation is conceivable in which a reduction in the effective free-stream velocity (i.e. by taking the resolved component in an oblique direction) would tend to destabilize such waves by putting the critical point in a more favourable part of the velocity profile. Nevertheless, this possibility does appear rather exceptional, and though we need to make some slight reservation on its account we can fairly confidently propose that the two-dimensional theory essentially covers the practical problem.

Our conclusions regarding the two main classes of waves—i.e. modified Tollmien–Schlichting waves, say Class A, and surface waves, say Class B—suggest two possible approaches to the design of practical measures for boundary-layer stabilization. The first is to specify the flexible medium with a fairly high over-all stiffness but such that, at values of k for the 'most dangerous' Tollmien–Schlichting waves, the velocity $c_0(k)$ coincides with the Tollmien-Schlichting wave velocity. Hence the response of the boundary will have a totally disruptive effect on the Tollmien-Schlichting mechanism of instability, the wall friction layer being effectively removed. In other words, Class A is turned into Class B, and the role of stabilization is taken over wholly by the internal damping of the flexible medium. The damping should be just large enough to prevent Class B waves from developing, and its amount is liable to be fairly critical; too much inhibits Class B instability yet tends to 'let in' Class A again, and might even result in an over-all destabilization because of its effect on Class A waves at velocities somewhat different from c_0; on the other hand, too little damping would have the disastrous effect of allowing the excitation of Class B waves, which would happen at Reynolds numbers less than the critical value for a rigid boundary since the damping action of viscosity is largely cancelled for Class B waves. It is probably important that $c_0(k)$ should be made nearly equal to—or perhaps just slightly greater than—the Tollmien–Schlichting wave velocity near the minimum of R on the neutral-stability curve for a rigid wall (e.g. for Blasius flow, $c = 0{\cdot}41U_0$ for $k = 0{\cdot}34/\delta^*$); this seems the first step to insure that the critical Reynolds number is raised to some extent by the present means. At higher Reynolds numbers the value of c and k for the Tollmien–Schlichting wave of greatest amplification steadily decrease, c being very roughly proportional to k, and it would seem that this relationship between c and k should be followed as closely and as far as possible in order to obtain maximum stabilization. If the stabilizing device were a layer of uniform visco-elastic solid material, for example, its thickness would have to be considerably less than a wavelength for c to vary with k as much as is required. A practical possibility is that such a layer might be 'tuned' to the Tollmien–Schlichting waves over a considerable range of Reynolds number (i.e. over a considerable length of the boundary layer) by varying its thickness.

The present method of design seems at first sight somewhat unpromising since its success apparently depends rather critically on the choice of properties for the flexible medium, and it may be expected to serve only for a very limited range of the free-stream velocity. However, the success of Krämer's experiments appears to be attributable to the particular mechanism of stabilization described here, and indeed a rough version of the general specifications explained above was established empirically by him. For instance, the flexible coatings used in his experiments were designed in such a way that our $c_0(k)$ would match the Tollmien–Schlichting wave velocity at the average Reynolds number of the coated area [in his paper (1960*b*) he tentatively assumes the boundary-layer waves to be equivalent in their effect on the boundary to standing pressure waves, and he specifies that the product $F\lambda$, where F is the frequency of resonance of the coating for a wavelength λ, should equal the representative velocity of the boundary-layer waves; the wavelength cancels from his design formula because his assumptions regarding the properties of the coating make it in effect non-dispersive]. His experimental results also showed that there is an optimum amount of damping in relation to stabilizing effect.

As the second possible method, the flexible medium might be designed with fairly low over-all stiffness but such that c_0 is large enough to avoid the possibility of Class B instability entirely. Stabilization of Class A waves is then effected by the compliant response of the boundary, i.e. as the result of a negative real α.

As the essential physical mechanism of stabilization in this case, the wall friction layer is modified so as to render Class A waves incapable of developing sufficient Reynolds stress to overcome viscous dissipation; in other words, there is a cancellation of the destabilizing effect of viscosity for which Prandtl's explanation was recalled in § 2. If this method is used, an important requirement is that damping in the medium should be small, because it has been seen in § 3 that damping provides an alternative means for the development of Reynolds stress by Class A waves and is therefore a destabilizing factor.

Roughly speaking, this second method requires an elastic medium which is both soft (to have a large response to surface pressures) and also light (to keep c_0 fairly high despite the first property), and which at the same time suffers little internal friction. The stiffness of the boundary may possibly need to be kept above a certain margin, however, so that the Kelvin–Helmholtz type of instability is avoided, and this factor may limit the ultimate scope of the method; this point deserves further study.

To obtain a useful stabilizing effect by the second method, it seems likely that the elastic constants of the medium (rigidity modulus, etc.) would have to be of about the same order of magnitude as the dynamic pressure $\frac{1}{2}\rho U_0^2$ of the free stream, and so the method might prove impracticable for low-speed applications because effective materials would be too fragile (though, of course, sufficiently enlarged clear compliances might possibly be obtained by use of cellular structures like Krämer's device). On the other hand, there may well be suitable materials for use at high speeds, and it is such applications that appear to offer the most exciting possibilities for this new means of boundary-layer stabilization.

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